Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

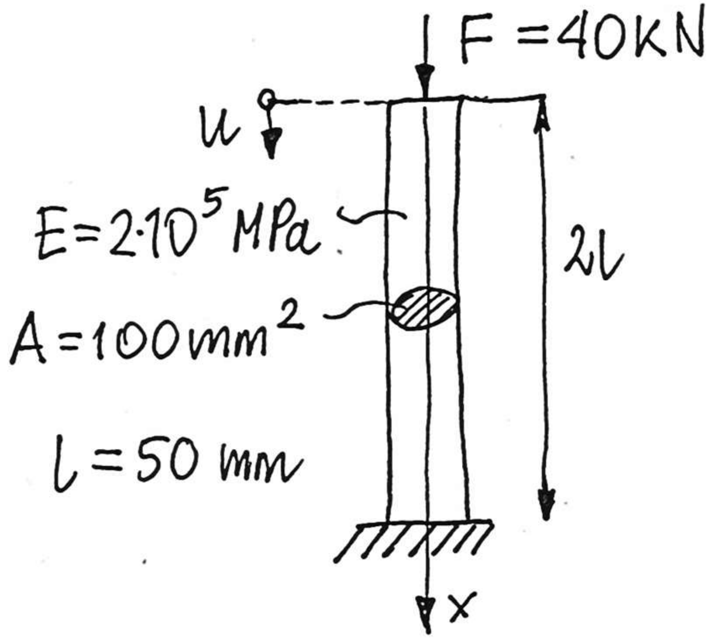
Finite element method 2 (FEM 2)

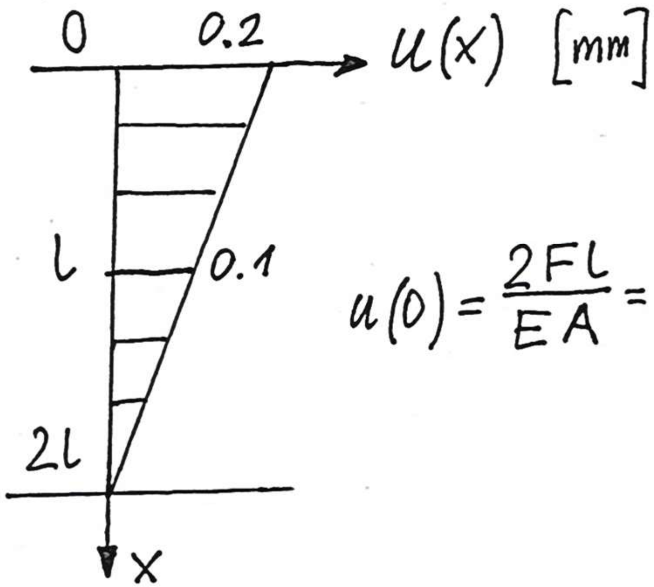
Contact problem

11.2021

CONTACT PROBLEM

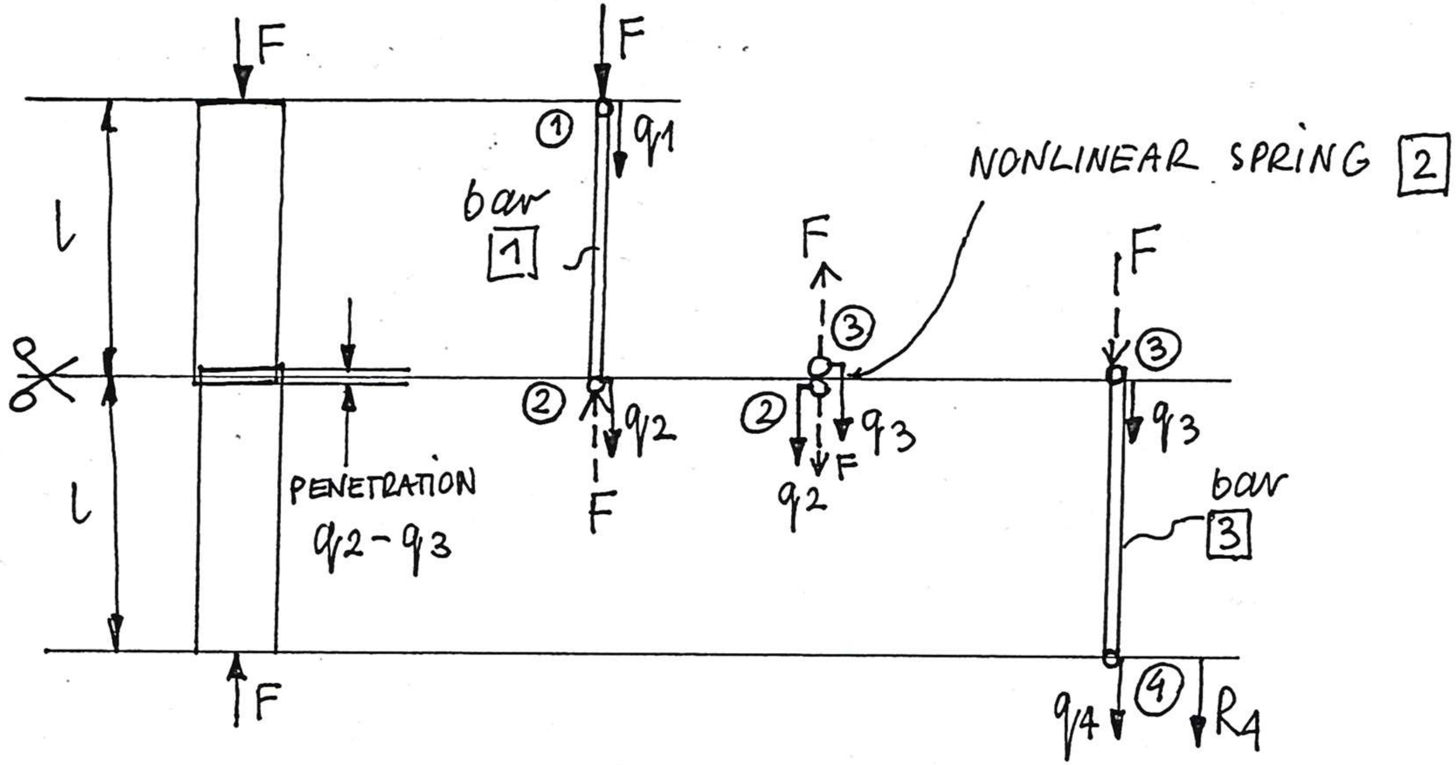
1°) LINEAR SOLUTION OF A COLUMN





$$u(0) = \frac{2Fl}{EA} = 0.2\,mm$$

2°) NONLINEAR SOLUTION (COLUMN DIVIDED INTO TWO PARTS)

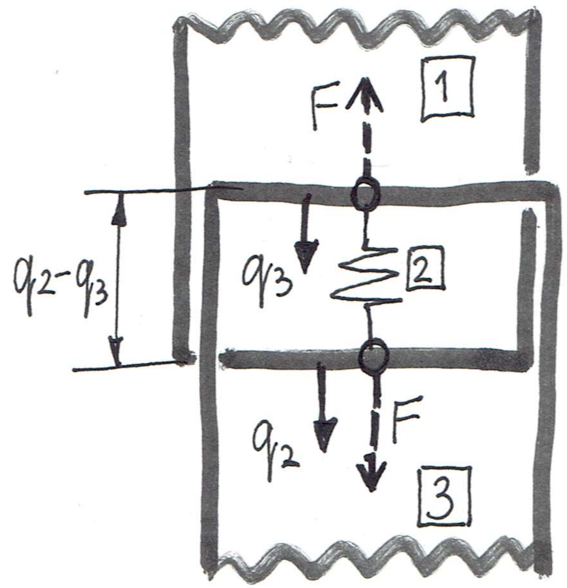


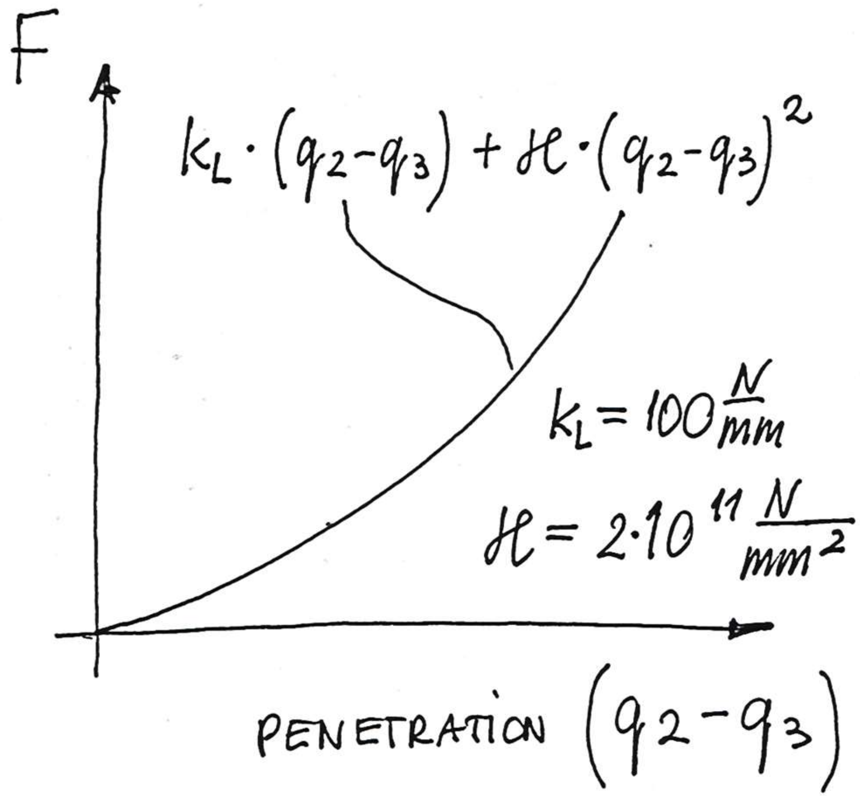
A SPRING IS ADDED TO STOP THE PENETRATION.

THE CONTACT AREA IS USUALLY A NONLINEAR FUNCTION OF LOAD. IN THE EXAMPLE THE AREA IS CONSTANT, SO INSTEAD WE ASSUME THAT THE SPRING HAS A NONLINEAR CHARACTERISTICS.

FORCE IN THE SPRING :

$$F = k \cdot (q_2 - q_3) \quad ; \quad \text{where} \quad k = k_L + \mathcal{H}(q_2 - q_3)$$





global vector of nodal parameters:

$$\underset{1\times 4}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4 \rfloor$$

global load vector:

$$\underset{1\times 4}{\lfloor F \rfloor} = \lfloor F, 0, 0, R_4 \rfloor$$

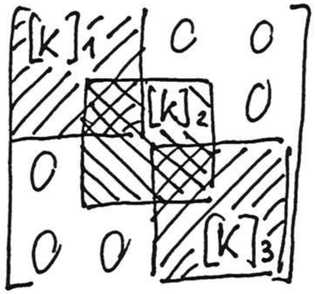
local stiffness matrices:

$$\underset{2\times 2}{[K]_1} = \underset{2\times 2}{[K]_3} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix} \quad ; \quad a = \frac{EA}{l} = 4\cdot 10^5 \frac{N}{mm}$$

$$\underset{2\times 2}{[K(q)]_2} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad ; \quad k = k_L + \varkappa (q_2 - q_3)$$

global stiffness matrix :

$$\underset{4\times4}{\left[K(q)\right]} =$$



$$= \begin{bmatrix} a & -a & 0 & 0 \\ -a & a+k & -k & 0 \\ 0 & -k & a+k & -a \\ 0 & 0 & -a & a \end{bmatrix}$$

set of nonlinear equations :

$$\underset{4\times4}{\left[K(q)\right]} \cdot \underset{4\times1}{\{q\}} = \underset{4\times1}{\{F\}}$$

boundary condition $q_4 = 0 \Rightarrow N = NDOF - NOF = 4 - 1 = 3$

$$\underset{3\times3}{\left[K(q)\right]} \cdot \underset{3\times1}{\{q\}} = \underset{3\times1}{\{F\}}$$

$$\begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad k = f(q_2, q_3)$$

(Newton – Raphson method)

tangent matrix :

$$\underset{3\times 3}{\left[K_T\right]} = \frac{\partial \underset{3\times 3}{\{F\}}}{\partial \underset{1\times 3}{\lfloor q \rfloor}} = \left[\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_1} , \frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_2} , \frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_3} \right]$$

$$\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_1} = \frac{\partial\left(\underset{3\times 3}{[K(q)]}\cdot\underset{3\times 1}{\{q\}}\right)}{\partial q_1} = \underbrace{\frac{\partial \underset{3\times 3}{[K(q)]}}{\partial q_1}}_{\underset{3\times 3}{[0]}}\cdot\underset{3\times 1}{\{q\}} + \underset{3\times 3}{[K(q)]}\cdot\frac{\partial \underset{3\times 1}{\{q\}}}{\partial q_1} = \underset{3\times 3}{[K(q)]}\cdot\begin{Bmatrix}\frac{\partial q_1}{\partial q_1}\\ \frac{\partial q_2}{\partial q_1}\\ \frac{\partial q_3}{\partial q_1}\end{Bmatrix} =$$

$$= \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix}\cdot\begin{Bmatrix}1\\0\\0\end{Bmatrix} = \begin{Bmatrix}a\\-a\\0\end{Bmatrix} \quad \text{1st row of } [K]$$

$$\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_2} = \frac{\partial\left(\underset{3\times 3}{[K(q)]}\cdot\underset{3\times 1}{\{q\}}\right)}{\partial q_2} = \frac{\partial \underset{3\times 3}{[K(q)]}}{\partial q_2}\cdot \underset{3\times 1}{\{q\}} + \underset{3\times 3}{[K(q)]}\cdot \frac{\partial \underset{3\times 1}{\{q\}}}{\partial q_2} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial k}{\partial q_2} & -\frac{\partial k}{\partial q_2} \\ 0 & -\frac{\partial k}{\partial q_2} & \frac{\partial k}{\partial q_2} \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \underset{3\times 3}{[K(q)]} \cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_2} \\ \frac{\partial q_2}{\partial q_2} \\ \frac{\partial q_3}{\partial q_2} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varkappa & -\varkappa \\ 0 & -\varkappa & \varkappa \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ \varkappa(q_2-q_3) \\ -\varkappa(q_2-q_3) \end{Bmatrix} + \underbrace{\begin{Bmatrix} -a \\ a+k \\ -k \end{Bmatrix}}_{\text{2nd row of } [K]} = \begin{Bmatrix} -a \\ a+k_L+2\varkappa(q_2-q_3) \\ -(k_L+2\varkappa(q_2-q_3)) \end{Bmatrix}$$

$$\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_3} = \frac{\partial \left(\underset{3\times 3}{[K(q)]} \cdot \underset{3\times 1}{\{q\}} \right)}{\partial q_3} = \frac{\partial \underset{3\times 3}{[K(q)]}}{\partial q_3} \cdot \underset{3\times 1}{\{q\}} + \underset{3\times 3}{[K(q)]} \cdot \frac{\partial \underset{3\times 1}{\{q\}}}{\partial q_3} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial k}{\partial q_3} & -\frac{\partial k}{\partial q_3} \\ 0 & -\frac{\partial k}{\partial q_3} & \frac{\partial k}{\partial q_3} \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \underset{3\times 3}{[K(q)]} \cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_3} \\ \frac{\partial q_2}{\partial q_3} \\ \frac{\partial q_3}{\partial q_3} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\varkappa & \varkappa \\ 0 & \varkappa & -\varkappa \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ -\varkappa(q_2-q_3) \\ \varkappa(q_2-q_3) \end{Bmatrix} + \underbrace{\begin{Bmatrix} 0 \\ -k \\ a+k \end{Bmatrix}}_{\text{3rd row of } [K]} = \begin{Bmatrix} 0 \\ -(k_L + 2\varkappa(q_2-q_3)) \\ a + k_L + 2\varkappa(q_2-q_3) \end{Bmatrix}$$

Thus :

$$\underset{3\times 3}{[K_T]} = \begin{bmatrix} a & -a & 0 \\ -a & a + k_L + 2\varkappa(q_2 - q_3) & -(k_L + 2\varkappa(q_2 - q_3)) \\ 0 & -(k_L + 2\varkappa(q_2 - q_3)) & a + k_L + 2\varkappa(q_2 - q_3) \end{bmatrix}$$

initial solution :

$$\underset{3\times 1}{\{q\}}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \underset{3\times 3}{[K(q)]}_0 = \begin{bmatrix} a & -a & 0 \\ -a & a+k_L & -k_L \\ 0 & -k_L & a+k_L \end{bmatrix}$$

$$\underset{3\times 3}{[K_T]}_1 = \frac{\partial \underset{N\times 1}{\{F\}}}{\partial \underset{1\times 3}{\lfloor q \rfloor}_0} = \underset{3\times 3}{[K(q)]}_0$$

SOLUTION AT ITERATION „ i ":

$$\underset{N\times N}{\left[K_T\right]_i} = \frac{\partial \underset{N\times 1}{\{F\}}}{\partial \underset{1\times N}{\lfloor q \rfloor}_{i-1}} = \begin{bmatrix} a & -a & 0 \\ -a & a+k_L+2\varkappa(q_2-q_3) & -(k_L+2\varkappa(q_2-q_3)) \\ 0 & -(k_L+2\varkappa(q_2-q_3)) & a+k_L+2\varkappa(q_2-q_3) \end{bmatrix}_{i-1}$$

$$\underset{3\times 1}{\{R\}_i} = \underset{3\times 1}{\{F\}} - \underset{3\times 3}{\left[K(q)\right]_{i-1}} \cdot \underset{3\times 1}{\{q\}_{i-1}}$$

$$\underset{3\times 1}{\{\Delta q\}_i} = \left[K_T\right]^{-1}_{i-1} \cdot \underset{3\times 1}{\{R\}_i} \qquad (i = 1, \dots, n)$$

$$\underset{3\times 1}{\{q\}_i} = \underset{3\times 1}{\{q\}_{i-1}} + \underset{3\times 1}{\{\Delta q\}_i}$$

convergence criteria :

displacement criterion :

$\varepsilon = 0.0005$

$$U2_{NORM_i} = \frac{\| \underset{3\times1}{\{\Delta q\}_i} \|_2}{\| \underset{3\times1}{\{q\}_i} \|_2} \leqslant \varepsilon$$

force criterion :

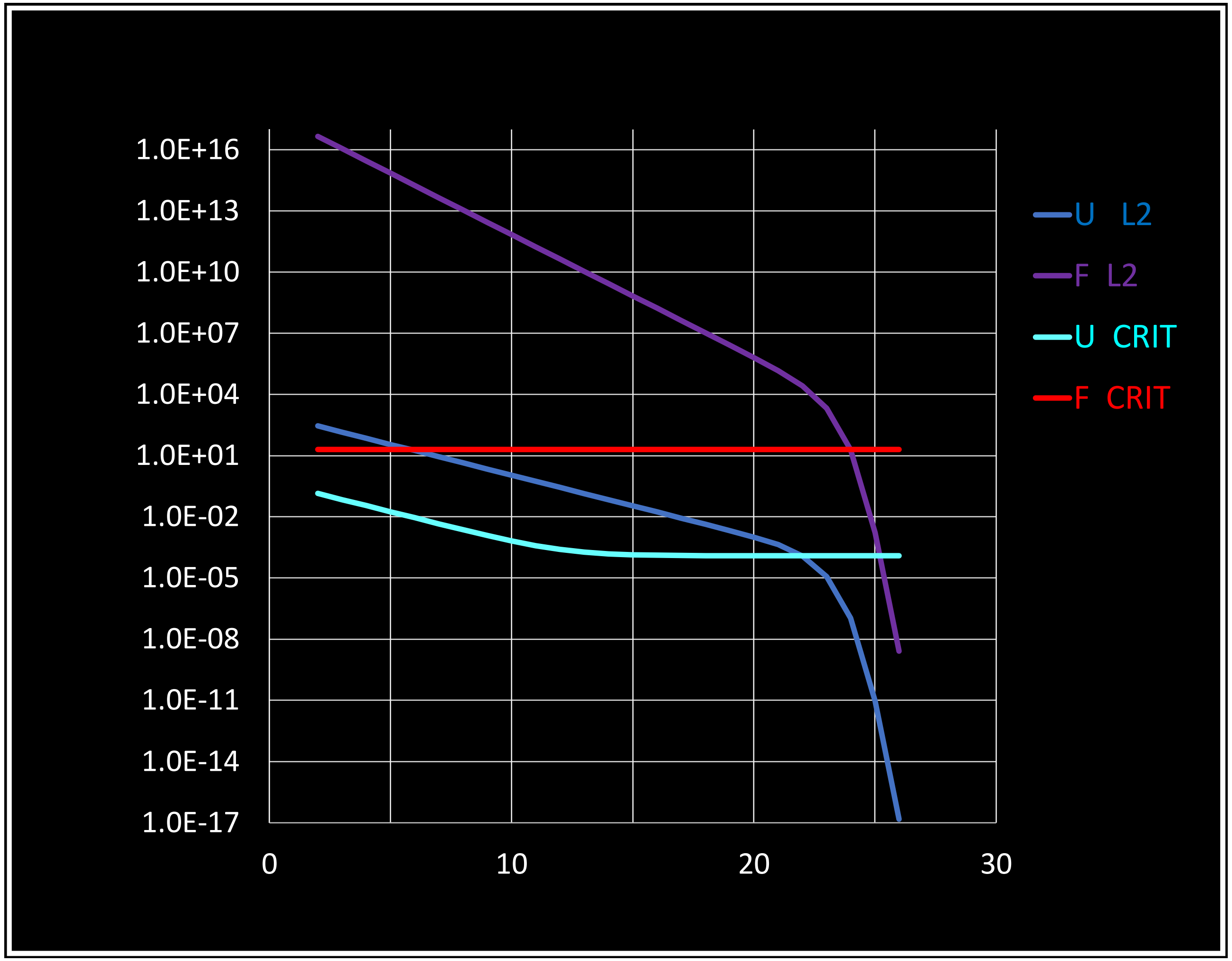
$\delta = 0.0005$

$$F2_{NORM_i} = \frac{\| \underset{3\times1}{\{R\}_i} \|_2}{\| \underset{3\times1}{\{F\}} \|_2} \leqslant \delta$$

FORCE IN THE SPRING : (CONTACT ELEMENT)

$$F_{c_i} = \Big(\big(k_L + \mathcal{H}(q_2 - q_3)\big) \cdot (q_2 - q_3) \Big)_i =$$

$$= \left(k_L + \mathcal{H} \cdot \text{PENETRATION}\right) \cdot \text{PENETRATION}$$



1.0E+16
1.0E+13
1.0E+10
1.0E+07
1.0E+04
1.0E+01
1.0E-02
1.0E-05
1.0E-08
1.0E-11
1.0E-14
1.0E-17
0
10
20
30
U L2
F L2
U CRIT
F CRIT